THE FORMULATION OF VARIATIONAL PROBLEMS OF GAS DYNAMICS

(K POSTANOVKE VARIATSIONNYKH ZADACH GAZOVOI DINAMIIKI)

PMM Vol.27, No.1, 1963, pp. 183-184

V.M. BORISOV and Iu.D. SHMYGLEVSKII
(Moscow)

(Received October 10, 1962)

The equations of gas dynamics of two dimensional isoenergetic and isentropic flows, expressed by contour integrals, have the form

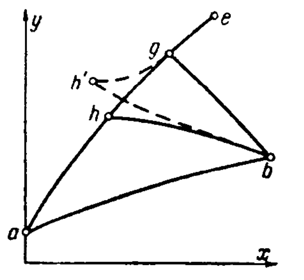
$$\oint f(y, w, \vartheta)\,dx - \varphi(y, w, \vartheta)\,dy = 0 \qquad \begin{pmatrix} w = w(x, y) \\ \vartheta = \vartheta(x, y) \end{pmatrix} \qquad (1)$$
$$\oint F(y, w, \vartheta)\,dx - \Phi(y, w, \vartheta)\,dy = 0$$

where x, y are the Cartesian coordinates in the plane of the flow under consideration; w, ϑ are respectively the modulus of the velocity and the angle of inclination of the velocity to the x-axis. The integration is carried out along the arbitrary closed contour L. The first of equations (1) can be looked upon as the equation of motion projected on the x-axis, the second as an equation of continuity.

In differential form equations (1) have the form

$$\frac{\partial\varphi}{\partial x} + \frac{\partial f}{\partial y} = 0, \qquad \frac{\partial\Phi}{\partial x} + \frac{\partial F}{\partial y} = 0 \qquad (2)$$

One of the possible formulations of the problems concerning the search for bodies with minimum wave drag (or for nozzles with maximum thrust) is the following. The points a and b and the characteristic curve ae of the inflowing stream are given (see figure). The control contour ahb is introduced, where hb is the characteristic curve of equations (2), arriving at the point b. Equations (1) allow one to register the magnitude of the resistance χ and the zero mass flux of gas Ψ through the contour ab in the form



$$\chi = \int_a^h f dx - \varphi dy + \int_h^b f dx - \varphi dy \tag{3}$$

$$\Psi = 0 = \int_a^h F dx - \Phi dy + \int_h^b F dx - \Phi dy \tag{4}$$

For the solution of a problem it is necessary to find a function w on hb, which has an extreme of the function (3) under conditions (4) and

$$X \equiv x_b - x_a = \int_a^h dx + \int_h^b dx \tag{5}$$

taking into account the continuity of functions w and ϑ at the point h and the equations of the characteristic curve, of which the first gives the relation between dx and dy, while the second is the so-called compatibility condition.

Such a formulation of the variational problems takes into account all relations on hb and is used in [1-3].

Rao [4] proposed another approach to the formulation of these problems. An arbitrary line is at first chosen as the closing line gb of the control contour agb, but the relation between w and ϑ on gb is not taken into account. This relation is specified by the following circumstances. Let the functions w and ϑ on gb be found by some means, for example by solution of the variational problem. Then the solution of the problems of Cauchy for equations (2) with initial values on gb defines a solution in the triangle $gh'b$. Thus, generally speaking, the located characteristic curve gh' does not coincide with the segment gh of the given characteristic curve ae. For this reason the formulation of the problem [4] is incomplete. Disregard of one of the relations in the general case leads to incorrect results. However, a full formulation of the problem is made difficult in a given case by the fact that the relation between w and ϑ on gb is unknown in an explicit form.

The adduced considerations are obvious, but the appearance of later papers [5,6], in which great attention is given to the method of Rao but the incorrectness of the formulation of the problem goes unnoticed, gives rise to the possibility of mistakes during the formulation of new problems. It should be mentioned that the basis of the method of Rao in [5] affects only the achievement of the conditions of compatibility on the located line gb with the characteristic direction.

We shall show that the correct end result in [4] was to a certain extent obtained accidentally. It is possible to rewrite equations (2) in the form

$$\begin{aligned} \varphi_w w_x + \varphi_\vartheta \vartheta_x + f_w w_y + f_\vartheta \vartheta_y + f_y &= 0 \\ \Phi_w w_x + \Phi_\vartheta \vartheta_x + F_w w_y + F_\vartheta \vartheta_y + F_y &= 0 \end{aligned} \tag{6}$$

where the partial derivatives are noted by indices. The equations of the characteristic curves of system (6) are determined by the equality

$$\begin{vmatrix} \varphi_w \tau - f_w & \varphi_\vartheta \tau - f_\vartheta \\ \Phi_w \tau - F_w & \Phi_\vartheta \tau - F_\vartheta \end{vmatrix} = 0 \qquad \left(\tau = \frac{dy}{dx}\right) \tag{7}$$

In the formulation [4] the variational problem is reduced to a search for the functions w and ϑ on gb, which yield an absolute extremum of the functional

$$I = \int_a^g [\lambda_1 (f - \varphi\tau) + \lambda_2 (F - \Phi\tau) + \lambda_3]\, dx + \int_g^b [\lambda_1 (f - \varphi\tau) + \lambda_2 (F - \Phi\tau) + \lambda_3]\, dx \tag{8}$$

where λ_1, λ_2, λ_3 are constant Lagrange multipliers. The integral along ag is a function of the upper boundary. Making a variation, we obtain, in particular, the equations of Euler on gb or on part of this line

$$\lambda_1 (f_w - \varphi_w \tau) + \lambda_2 (F_w - \Phi_w \tau) = 0, \quad \lambda_1 (f_\vartheta - \varphi_\vartheta \tau) + \lambda_2 (F_\vartheta - \Phi_\vartheta \tau) = 0 \tag{9}$$

The multipliers λ_1 and λ_2, generally speaking, are not equal to zero, therefore the condition of compatibility of equations (9) coincides with the equality (7) and unexpectedly gives an equation for determining the quantity $\tau = dy/dx$.

Let us now examine the variational problem, in which the condition (5) is replaced by a condition of a more general form, not connected with equations (1)

$$P = \int_a^g U(y, w, \vartheta)\, dx - V(y, w, \vartheta)\, dy + \int_g^b U(y, w, \vartheta)\, dx - V(y, w, \vartheta)\, dy$$

In this case the functional of type (8) is expressed in the form

$$\begin{aligned} J = &\int_a^g [\lambda_1 (f - \varphi\tau) + \lambda_2 (F - \Phi\tau) + \lambda_3 (U - V\tau)]\, dx + \\ &+ \int_g^b [\lambda_1 (f - \varphi\tau) + \lambda_2 (F - \Phi\tau) + \lambda_3 (U - V\tau)]\, dx \end{aligned}$$

The equations of Euler on the line gb or on part of it have the form

$$\lambda_1 (f_w - \varphi_w \tau) + \lambda_2 (F_w - \Phi_w \tau) + \lambda_3 (U_w - V_w \tau) = 0$$
$$\lambda_1 (f_\vartheta - \varphi_\vartheta \tau) + \lambda_2 (F_\vartheta - \Phi_\vartheta \tau) + \lambda_3 (U_\vartheta - V_\vartheta \tau) = 0$$

and do not determine τ. If the characteristic line is not chosen as *gb* then the latter equations, as already mentioned, give results which contradict the data given along the characteristic curve *ae*.

Let us imagine at last, ideally, that in the formulation of an original problem the relation between w and ϑ on *gb* is learned. In this case the expression under the integral along *gb* in the equality (8) will have an additional component. The corresponding equations of Euler cease to be homogeneous relative to λ_1 and λ_2. It turns out that the additional limitation on τ of type (7) in this case reveals no contradiction with the full formulation of the problem. This result can also be applied to the case of non-isentropic currents.

BIBLIOGRAPHY

1. Nikol'skii, A.A., O telakh vrashcheniia s protokom, obladaiushchikh naimen'shim volnovym soprotivleniem v sverkhzvukovom potoke (On ducted bodies of revolution, posessing least wave resistance in a supersonic stream). *Sb. teoret. rabot. po aerodinamike*, pp.56-63. Oborongiz, 1957.

2. Guderley, G. and Hantsch, E., Beste Formen für achsensymmetrische Überschallschubdüsen. *Z. Flugwissenschaften* Vol. 3, No. 9, pp. 305-313, September 1955.

3. Shmyglevskii, Iu.D., Nekotorye variatsionnye zadachi gazovoi dinamiki osesimmetrichnykh sverkhzvukovykh techenii (Some variational problems of gas dynamics of axisymmetric supersonic flows). *PMM* Vol. 21, No. 2, 1957.

4. Rao, G.V.R., Exhaust nozzle contour for optimum thrust. *Jet Propulsion* Vol. 28, No. 6, pp. 377-382, 1958.

5. Guderley, G., On Rao's method for the computation of exhaust nozzles. *Z. Flugwissenschaften* Vol. 7, No. 12, pp. 345-350, December, 1959.

6. Guderley, K.G., Armitage, J.V. and Valentine, E.M., Nose and inlet shapes of minimum drag in supersonic flow. *IAS Paper* No. 62-116, 1962.

Translated by C.McF.